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On bi-Hamiltonian geometry of the Lagrange top

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Abstract
We consider three different incompatible bi-Hamiltonian structures for the Lagrange top, which have the same foliation by symplectic leaves. These bivectors may be associated with different 2-coboundaries in the Poisson–Lichnerowicz cohomology defined by canonical bivector on $e^*(3)$.

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1. Introduction

In recent years considerable progress has been made in investigations of the integrable systems having enough functionally independent integrals of motion in the involution with respect to a pair of compatible Poisson brackets on the bi-Hamiltonian manifold M:

$$\{H_i, H_j\} = \{H_i, H_j\}' = 0, \qquad i, j = 1, \dots, n. \tag{1.1}$$

Following [17] such systems will be called bi-integrable systems.

Historically the majority of them come from stationary flows, restricted flows or the Lax equations of underlying soliton systems (see references in [1, 9]). Construction of integrals of motion for such systems is usually based on the Lenard–Magri recurrence relations. In order to solve the corresponding equations of motion in framework of the separation of variables method we have to use some suitable reductions of the Poisson bivectors [1, 10].

The other class of bi-integrable systems comes from r-matrix algebras, classifications of 2-coboundaries in the Poisson–Lichnerowicz cohomology and the separation of variables method [3, 18–21]. The corresponding Poisson brackets $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}'$ have a common foliation as their symplectic leaf foliations. In this case we lose benefits given by bi-Hamiltonian recurrence relations, but we can obtain the separated variables directly.

The main aim of this paper is to discuss different bi-Hamiltonian structures of both types for the Lagrange top.

2. The bi-Hamiltonian manifolds

In this section we describe the manifolds where our bi-integrable systems will be defined.

Let M be a finite-dimensional Poisson manifold endowed with a bivector P fulfilling the Jacobi condition

$$[P, P] = 0$$

with respect to the Schouten bracket $[\cdot, \cdot]$. We will suppose that P has the constant corank k, $\dim M = 2n + k$, and that $C_1, \dots, C_k$ are globally defined independent Casimir functions on M

$$P dC_a = 0, \qquad a = 1, \dots, k.$$

The $2n$-dimensional symplectic leaves of P form a symplectic foliation.

A bi-Hamiltonian manifold M is a smooth (or complex) manifold endowed with two compatible bivectors P, P' such that

$$[P, P] = [P, P'] = [P', P'] = 0. \tag{2.1}$$

Classification of compatible Poisson bivectors on low-dimensional Poisson manifolds is nowadays a subject of intense research. However the higher dimensional problem is virtually untouched.

2.1. Integrals of motion from the Poisson bivectors

Let us consider bi-Hamiltonian manifold M with some known bivectors P and P'. Moreover, let us suppose that there are k polynomial Casimir functions of the Poisson pencil $P_\lambda = P' - \lambda P$,

$$H^a(\lambda) = \sum_{i=0}^{n_a} H_i^a \lambda^{n_a - i}, \qquad H_0^a = C_a, \qquad a = 1, \dots, k, \tag{2.2}$$

such that $n_1 + n_2 + \cdots + n_k = n$ and the differentials of the coefficients H_i^a are linearly independent of M.

The collection of the n bi-Hamiltonian vector fields

$$X_i^{(a)} = P\, \mathrm{d}H_i^a = P'\, \mathrm{d}H_{i-1}^a, \qquad i = 1, \dots, n_a, \qquad a = 1, \dots, k \tag{2.3}$$

associated with the Lenard–Magri sequences defined by the Casimirs $H^a(\lambda)$ is called the Gel'fand–Zakharevich system.

The standard arguments from the theory of Lenard–Magri chains show that all the coefficients H_i^a (2.2) pairwise commute with respect to both $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}'$. It allows us to get non-trivial bi-integrable systems with integrals of motion H_i^a starting from the Casimir functions $H_0^a = C_a$ only.

If there exists a foliation of M, transversal to the symplectic leaves of P and compatible with the Poisson pencil (in a suitable sense), then the restrictions of the Gel'fand–Zakharevich systems on symplectic leaves of P are separable in the so-called Darboux–Nijenhuis variables [1].

Summing up, if we have two compatible Poisson bivectors P, P' and the Casimir functions C_a of P, then we can get integrals of motion H_i in the bi-involution using recurrence relations (2.3) and, if we are lucky, then we obtain the separated variables after some appropriate reduction.

Remark 1. We know such Poisson structures for the Clebsch problem, for the Steklov–Lyapunov model, for the Kowalewski gyrostat but the suitable reduction procedures are unknown up to now.

2.2. The Poisson bivectors from integrals of motion

Let us consider bi-Hamiltonian manifold M with canonical bivector P and some integrable system with integrals of motion H_m. According to the Liouville–Arnold theorem any integrable system admits separation of variables in the action-angle coordinates.

Integrable system on the symplectic leaves of M will be said to be separable if the complete integral $S(q,t)$ of the Hamilton–Jacobi equation has an additive form

$$S(q,t,\alpha_1,\dots,\alpha_n,C_1,\dots,C_k) = -Ht + \sum_{i=1}^{n} S_i(q_i,\alpha_1,\dots,\alpha_n,C_1,\dots,C_k) \tag{2.4}$$

in a set of canonical variables $(p,q) = (p_1,\dots,p_n,q_1,\dots,q_n)$. In this case the Jacobi equations

$$p_j = \frac{\partial}{\partial q_j} S_j(q_j,\alpha_1,\dots,\alpha_n,C_1,\dots,C_k) \tag{2.5}$$

yield the separated relations [1, 13]:

$$\phi_j(p_j,q_j,\alpha_1,\dots,\alpha_n,C_1,\dots,C_k) = 0, \qquad \det\left[\frac{\partial\phi_i}{\partial\alpha_j}\right] \neq 0, \qquad \{q_i,p_i\} = 1. \tag{2.6}$$

Separated equations depending on additional parameters $C_1,\dots,C_k$ are well known, as an example we could allege on such well-known integrable systems as the Jacobi problem of geodesics on ellipsoid, the Kowalevski top, the Goryachev–Chaplygin top, the Steklov–Lyapunov model, the Heisenberg and Gaudin magnets and so on, see [1, 13] and references within.

Remark 2. The function $S(q,t,\alpha_1,\dots,\alpha_n,C_1,\dots,C_k)$ is the generating function of the canonical transformation from the separated variables (q,p) to the action-angle variables. The modern proof of the action-angle theorem in the general, and most natural, context of integrable systems on the Poisson manifolds may be found in [7].

If we resolve the separated equations (2.5)–(2.6) with respect to parameters $\alpha_1,\dots,\alpha_n$ one gets n independent integrals of motion

$$\alpha_m = H_m(p,q,C), \qquad m = 1,\dots,n, \tag{2.7}$$

as functions on the phase space M with coordinates $z = (p,q,C)$.

These integrals of motion $H_i(p,q,C)$ are in the involution

$$\{H_i,H_j\}_f = 0, \qquad i,j = 1,\dots,n, \tag{2.8}$$

with respect to the following bracket $\{\cdot,\cdot\}_f$ on M,

$$\begin{aligned} &\{q_i,p_j\}_f = \delta_{ij} f_j(p_j,q_j), \\ &\{p_i,p_j\}_f = \{q_i,q_j\}_f = \{p_i,C_j\}_f = \{q_i,C_j\}_f = \{C_i,C_j\}_f = 0, \end{aligned} \tag{2.9}$$

which depends on n arbitrary functions $f_1(p_1,q_1),\dots,f_n(p_n,q_n)$ [17]. This bracket defines the Poisson bivector

$$P^f = \begin{pmatrix} 0 & \mathrm{diag}(f_1,\dots,f_n) & 0 \\ -\mathrm{diag}(f_1,\dots,f_n) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.10}$$

compatible with the Poisson bivector P on M and such that

$$P^f\,\mathrm{d}C_a = 0, \qquad a = 1,\dots,k. \tag{2.11}$$

If all the $f_i \neq 0$, then P^f has the same foliations by symplectic leaves as P. So, it is an explicit construction of the Poisson structures having the same foliation by symplectic leaves.

Compatibility conditions (2.1), equations (2.8) and (2.11) may be checked in any coordinate system on M. So, for any integrable system on M we can try to solve the following system of equations,

$$[P, P^f] = [P^f, P^f] = 0, \qquad P^f\, \mathrm{d}C_a = 0, \qquad \{H_m, H_l\}_f = 0, \tag{2.12}$$

with respect to P^f. Obviously enough, in their full generality equations (2.12) are too difficult to be solved because it has infinitely many solutions labeled by different separated coordinates and their functions $f = (f_1, \ldots, f_n)$ [17]. In order to get a particular solution of the system (2.12) we have to use some additional assumptions or the couple of ansätze [16, 20, 21].

Nevertheless, using any known solution P^f of (2.12), we can easily solve the following system of algebraic equations,

$$P^f\, \mathrm{d}H_m = P \sum_{l=1}^{n} F_{ml}\, \mathrm{d}H_l, \qquad m = 1, \ldots, n, \tag{2.13}$$

with respect to entries of the $n \times n$ control matrix F. According to [1] the eigenvalues of the control matrix F are the separated coordinates

$$\det(F - \lambda \mathrm{I}) = \prod_{j=1}^{n} (\lambda - q_j).$$

The solution of the equations (2.12) and (2.13) may be considered as a direct method of computation of the separated coordinates q_j starting with given integrals of motion only.

Remark 3. In fact we postulate in (2.7) and (2.9) that our separated variables are 'invariant' with respect to the Casimirs, as the one considered in [15].

Remark 4. Bivectors P' fulfilling the compatibility condition $[P, P'] = 0$ are called 2-cocycles in the Poisson–Lichnerowicz cohomology defined by P on M [8]. The Lie derivative of P along any vector field X on M,

$$P' = \mathcal{L}_X(P) \tag{2.14}$$

is 2-coboundary, i.e. it is a 2-cocycle associated with the Liouville vector field X. For such bivectors P' the compatibility conditions (2.1) are reduced to the single equation

$$[\mathcal{L}_X(P_0), \mathcal{L}_X(P_0)] = 0, \quad \Leftrightarrow \quad \left[\mathcal{L}_X^2(P_0), P_0\right] = 0. \tag{2.15}$$

The second Poisson–Lichnerowicz cohomology group $H^2_{P_0}(M)$ of M is precisely the set of bivectors P_1 solving $[P, P'] = 0$ modulo the solutions of the form $P_1 = \mathcal{L}_X(P_0)$. We can interpret $H^2_{P_0}(M)$ as the space of infinitesimal deformations of the Poisson structure modulo trivial deformations. For regular Poisson manifolds cohomology reflects the topology of the leaf space and the variation in the symplectic structure as one passes from one leaf to another.

In our case the components of the Liouville vector field X in the variables (p, q, C) are equal to

$$X_j = \begin{cases} F_j(q_j, p_j), & j = 1, \ldots, n \\ 0, & j = n+1, \ldots, 2n+k \end{cases}$$

and bivector $P^f = \mathcal{L}_X(P)$ has the form (2.10) with

$$f_j(q_j, p_j) = -\frac{\partial}{\partial q_j} F_j(q_j, p_j).$$

So, in fact in the separation of variables method we are looking for special 2-coboundaries (2.10) having the same foliation by symplectic leaves as the bivector P (2.11).

Summing up, if we have the canonical Poisson bivectors P, it is Casimir functions C_a and integrals of motion H_m for some integrable system, then we can try to get the compatible Poisson bivector P^f from equations (2.1), which immediately gives rise to the corresponding separated variables.

3. The Lagrange top

Let two vectors $J = (J_1, J_2, J_3)$ and $x = (x_1, x_2, x_3)$ be coordinates on the Euclidean algebra $e(3)^*$ with the Lie–Poisson bracket

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \qquad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \qquad \{x_i, x_j\} = 0, \tag{3.1}$$

where ε_{ijk} is the totally skew-symmetric tensor. This bracket has two Casimir functions

$$C_1 = |x|^2 \equiv \sum_{k=1}^{3} x_k^2, \qquad C_2 = (x, J) \equiv \sum_{k=1}^{3} x_k J_k. \tag{3.2}$$

Fixing their values one gets a generic symplectic leaf of $e(3)$

$$\mathcal{O}_{ab} : \quad \{x, J : C_1 = \alpha^2, C_2 = \beta\},$$

which is a four-dimensional symplectic manifold. As usual we identify $(\mathbb{R}^3, \wedge)$ and $(so(3), [\cdot, \cdot])$ by using the well- known isomorphism of the Lie algebras

$$z = (z_1, z_2, z_3) \to z_M = \begin{pmatrix} 0 & z_3 & -z_2 \\ -z_3 & 0 & z_1 \\ z_2 & -z_1 & 0 \end{pmatrix}, \tag{3.3}$$

where $\wedge$ is the cross product in $\mathbb{R}^3$ and $[\cdot, \cdot]$ is the matrix commutator in $so(3)$. In these coordinates the Poisson bivector on $e^*(3)$ is equal to

$$P = \begin{pmatrix} 0 & x_M \\ x_M & J_M \end{pmatrix}. \tag{3.4}$$

This Lie–Poisson tensor in (x, J) variables will be called canonical tensor and the transformations preserving this form of the Lie–Poisson tensor will be called canonical transformations.

We use this name because the rigid body motion about a fixed point under the influence of gravity is described by six canonical dynamical variables: three components of the angular momentum $J = (J_1, J_2, J_3)$ and three components of the gravity vector $x = (x_1, x_2, x_3)$, everything with respect to a moving orthonormal frame attached to the body.

The Lagrange top is one of the most classical examples of integrable systems with the following integrals of motion,

$$\widetilde{H}_1 = J_3, \qquad \widetilde{H}_2 = J_1^2 + J_2^2 + m J_3^2 + a x_3, \qquad a, m \in \mathbb{R}, \tag{3.5}$$

defined up to canonical transformations. This is a special case of rotation of a rigid body around a fixed point in a homogeneous gravitational field, characterized by the following conditions: the rigid body is rotationally symmetric, i.e. two of its three principal moments of inertia coincide, and the fixed point lies on the axis of rotational symmetry.

The explicit formulae for the position of the body in space were found by Jacobi [4]. For an actual integration of the corresponding equations of motion in terms of elliptic functions see [5] and for a more modern account [2, 11].

The geometric properties of integrable systems are invariant with respect to any transformation of the integrals of motion $H_k \to \widetilde{H}_k(H_1,\dots,H_n)$, which has an effect on the parametrization of trajectories of motion only. Therefore, when we search the variables of separation (2.4) or the second Poisson brackets (1.1) we can consider a more symmetric system with the following integrals of motion,

$$H_1 = \widetilde{H}_1 = J_3, \qquad H_2 = \widetilde{H}_2 - (m-1)\widetilde{H}_1 = J_1^2 + J_2^2 + J_3^2 + ax_3, \tag{3.6}$$

instead of the Lagrange top. We use this linear transformation of the initial integrals of motion (3.5) for the brevity only.

3.1. Recurrence relations

According to [2] the invariant manifold of the Lagrange top is isomorphic to the invariant manifolds of one-gap solutions of the nonlinear Schrödinger equation. Their bi-Hamiltonian structures may be identified as well.

So, for the Lagrange top there are two known Poisson bivectors P' compatible with the canonical bivector P (3.4) [2, 12]:

$$P_1' = \begin{pmatrix} 0 & x_3 & -x_2 & 0 & 0 & 0 \\ -x_3 & 0 & x_1 & 0 & 0 & 0 \\ x_2 & -x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{a}{2} & 0 \\ 0 & 0 & 0 & \frac{a}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P_2' = \begin{pmatrix} 0 & J_3 & -J_2 & 0 & \frac{a}{2} & 0 \\ -J_3 & 0 & J_1 & -\frac{a}{2} & 0 & 0 \\ J_2 & -J_1 & 0 & 0 & 0 & 0 \\ 0 & \frac{a}{2} & 0 & 0 & 0 & 0 \\ -\frac{a}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.7}$$

They are 2-coboundaries $P_{1,2}' = \mathcal{L}_{X_{1,2}}(P)$ and the corresponding Liouville vector fields $X_{1,2}$ may be obtained from the corresponding vector fields from [20, 21] by using contraction of $so^*(4)$ to $e^*(3)$.

The Poisson pencil $P_\lambda = P_1' - \lambda P$ has one non-trivial polynomial Casimir (2.2)

$$H^1(\lambda) = C_1, \qquad H^2(\lambda) = 2\lambda^2 C_2 + \lambda H_2 + aH_1,$$

while the second Poisson pencil $P_\lambda = P_2' - \lambda P$ has two non-trivial Casimirs

$$H^1(\lambda) = -2\lambda C_1 + H_2, \qquad H^2(\lambda) = -2\lambda C_2 + aH_1.$$

Using the corresponding recurrence relations

$$\begin{aligned} &0 = P_1'\,\mathrm{d}C_1, \\ &P_1' H_1 = 0, \qquad aP\,\mathrm{d}H_1 = P_1'\,\mathrm{d}H_2, \qquad P\,\mathrm{d}H_2 = 2P_1'\,\mathrm{d}C_2, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &P_2'\,\mathrm{d}H_2 = 0, \qquad P\,\mathrm{d}H_2 = -2P_2'\,\mathrm{d}C_1, \\ &P_2'\,\mathrm{d}H_1 = 0, \qquad aP\,\mathrm{d}H_1 = -2P_2'\,\mathrm{d}C_2, \end{aligned} \tag{3.9}$$

we can easily get integrals of motion $H_{1,2}$ (3.6) starting with the known Casimir functions $C_{1,2}$ (3.2).

The Poisson tensors P_1' and P_2' are compatible with each other, i.e. $[P_1', P_2'] = 0$. So, existence of the triad P, P_1', P_2' of mutually compatible Poisson bivectors leads to a tri-Hamiltonian structure for the Lagrange top. Reducing à la Marsden–Ratiu this tri-Hamiltonian structure we can get the separated variables for the Lagrange top. The reduction may not be unique, since possibly different separated variables can be constructed on the symplectic leaf of P [10].

3.2. Poisson structures with the same foliation by symplectic leaves

The generic solution of the equation $P_f dC_{1,2} = 0$ may be parametrized by two vector functions $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$,

$$P^f = \begin{pmatrix} (x, f)x_M & f \otimes (x \wedge J) + Q \\ -[f \otimes (x \wedge J) + Q]^T & -(x \wedge J)_3 g_M \end{pmatrix}, \tag{3.10}$$

where

$$Q = \begin{pmatrix} x_2(x \wedge g)_1 & x_2(x \wedge g)_2 & x_2(x \wedge g)_3 \\ -x_1(x \wedge g)_1 & -x_1(x \wedge g)_2 & -x_1(x \wedge g)_3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here $(u \otimes v)_{ij} = u_i v_j$ and $(u \wedge v)_j$ is the jth entry of the crossproduct $u \wedge v$ of two vectors u and v.

For any given integrable system on $e^*(3)$ functions f and g have to satisfy one algebraic equation

$$\{H_1, H_2\}_f = 0 \tag{3.11}$$

and the overdetermined system of algebro-differential equations (2.1). To solve these equations for the Lagrange top we use some hypothesis about the functions f.

3.2.1. Solution 1. If we put $f_3 = 0$ and $(x, f) = 0$, then one gets

$$f_1 = \frac{-\arccos\left(\frac{x_3}{|x|}\right)}{(x \wedge J)_3} x_2, \qquad f_2 = \frac{-\arccos\left(\frac{x_3}{|x|}\right)}{(x \wedge J)_3} x_1$$

and

$$g_1 = \frac{\arctan\left(\frac{x_1}{x_2}\right)}{(x \wedge J)_3} J_1, \qquad g_2 = \frac{\arctan\left(\frac{x_1}{x_2}\right)}{(x \wedge J)_3} J_2,$$

$$g_3 = \frac{-\arccos\left(\frac{x_3}{|x|}\right)}{(x \wedge J)_3} J_3 + \frac{x_3\left(\arccos\left(\frac{x_3}{|x|}\right) - \arctan\left(\frac{x_1}{x_2}\right)\right)(x_1 J_1 + x_2 J_2)}{\left(x_1^2 + x_2^2\right)(x \wedge J)_3}.$$

The corresponding bivector (3.10) we designate as P_1^f. In this case

$$F = \begin{pmatrix} \arctan\left(\frac{x_1}{x_2}\right) & 0 \\ 2\left[\arctan\left(\frac{x_1}{x_2}\right) - \arccos\left(\frac{x_3}{|x|}\right)\right]\left(J_3 - \frac{x_3(x_1 J_2 + x_2 J_2)}{x_1^2 + x_2^2}\right) & \arccos\left(\frac{x_3}{|x|}\right) \end{pmatrix}.$$

The eigenvalues of F are the separated variables

$$q_1 = \arctan\left(\frac{x_1}{x_2}\right), \qquad q_2 = \arccos\left(\frac{x_3}{|x|}\right),$$

which coincide with the Euler angles ϕ and θ, respectively. The canonically conjugated momenta read as

$$p_1 = -J_3, \qquad p_2 = -J_1 \cos\left(\arctan\left(\frac{x_1}{x_2}\right)\right) + J_2 \sin\left(\arctan\left(\frac{x_1}{x_2}\right)\right).$$

In variables (q, p, C) two compatible bivectors P and P_1^f have the standard form

$$P = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_1^f = \begin{pmatrix} 0 & \mathrm{diag}(q_1, q_2) & 0 \\ -\mathrm{diag}(q_1, q_2) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.12}$$

Using variables (q, p, C) we can easily prove that projections of the linear bivectors P_1' and P_2' (3.7) cannot be associated with the Euler angles.

3.2.2. Solution 2. If we put $f_3 = 0$ and $(x, f) \neq 0$ then one gets

$$f_1 = \frac{x_1 - \mathrm{i}x_2}{(x \wedge J)_3} J_2 + \frac{x_1 - \mathrm{i}x_2}{(J_1 - \mathrm{i}J_2)^2 |x|}, \qquad f_2 = -\frac{x_1 - \mathrm{i}x_2}{(x \wedge J)_3} J_1 - \frac{x_1 - \mathrm{i}x_2}{(J_1 - \mathrm{i}J_2)^2 |x|}$$

and

$$g_m = -\frac{J_1 - \mathrm{i}J_2}{(x \wedge J)_3} J_m.$$

The corresponding bivector (3.10) we designate as P_2^f. In this case

$$F = \begin{pmatrix} 0 & -\frac{\mathrm{i}}{2(x_2 + \mathrm{i}x_3)} \\ a & -\frac{J_2 + \mathrm{i}J_3}{x_2 + \mathrm{i}x_3} \end{pmatrix}$$

and one gets complex separated variables

$$q_{1,2} = -\frac{J_2 + \mathrm{i}J_3 \pm \sqrt{(J_2 + \mathrm{i}J_3)^2 - 2\mathrm{i}a(x_2 + \mathrm{i}x_3)}}{2(x_2 + \mathrm{i}x_3)}, \qquad \mathrm{i}^2 = -1.$$

In variables (q, p, C) two compatible bivectors P and P_2^f have the form (2.10)

$$P = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_2^f = \begin{pmatrix} 0 & -\mathrm{diag}\left(\frac{1}{q_1}, \frac{1}{q_2}\right) & 0 \\ \mathrm{diag}\left(\frac{1}{q_1}, \frac{1}{q_2}\right) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

First, these separated variables have been appear in the framework of the Sklyanin method and then have been recovered in [10] by reduction of the compatible linear Poisson bivectors P_1' and P_2' (3.7). In variables (q, p, C) these linear bivectors look like

$$P_1' = \begin{pmatrix} 0 & \mathrm{diag}\left(\frac{a}{2q_1}, \frac{a}{2q_2}\right) & w_1 \\ -\mathrm{diag}\left(\frac{a}{2q_1}, \frac{a}{2q_2}\right) & 0 & w_2 \\ -w_1 & -w_2 & 0 \end{pmatrix}$$

and

$$P_2' = \begin{pmatrix} 0 & -\mathrm{diag}\left(\frac{a^2}{4q_1^2}, \frac{a^2}{4q_2^2}\right) & w_3 \\ \mathrm{diag}\left(\frac{a^2}{4q_1^2}, \frac{a^2}{4q_2^2}\right) & 0 & w_4 \\ -w_3 & -w_4 & 0 \end{pmatrix}.$$

The matrix elements of w_k are brackets $\{q_i, C_j\}_m'$ and $\{p_i, C_j\}_m'$, which are some non-trivial rational functions on the separated variables and Casimirs. For instance

$$\{q_1, C_2\}_1' = \frac{\mathrm{i}q_1 p_2}{q_1 - q_2}.$$

We can see that reductions of P_1' and P_2^f on symplectic leaf of P are identical up to multiplication on a constant. Roughly speaking, in this case reduction consists of removing the last rows and the last columns of P_1' and P_2^f.

Remark 5. For the Lagrange top this reduction procedure leads to the complex separated variables and the complex separated relations, which are useless for the qualitative analysis of motion.

3.2.3. Solution 3. If we put $f_1 = 0$ then one gets $(x, f) = 1$ and

$$g_m = \frac{ax_m}{2(x\wedge J)_3(x_3 - \mathrm{i}x_2)}, \qquad f_2 = f_3 = \frac{1}{x_2 + \mathrm{i}x_3}. \tag{3.13}$$

The corresponding bivector (3.10) we designate as P_3^f. In this case

$$F = \begin{pmatrix} J_1 - \mathrm{i}J_2 & 0 \\ a(x_1 - \mathrm{i}x_2) + \frac{a(x_1-\mathrm{i}x_2)^2}{|x|(J_1-\mathrm{i}J_2)^2} & -\frac{x_1-\mathrm{i}x_2}{|x|(J_1-\mathrm{i}J_2)} \end{pmatrix}$$

and the separated coordinates are

$$q_1 = J_1 - \mathrm{i}J_2, \qquad q_2 = -\frac{x_1 - \mathrm{i}x_2}{|x|(J_1 - \mathrm{i}J_2)}.$$

In variables (q, p, C) two compatible bivectors P and P_3^f have the form (3.12) and, therefore, P_3^f is 2-coboundary. The corresponding separated equations, the Lax matrices, the r-matrix formalism and the Bäcklund transformations could be found in [6].

The Poisson bivectors P_k^f, $k = 1, 2, 3$, are incompatible with each other, i.e. $\left[P_i^f, P_j^f\right] \neq 0$ at $i \neq j$. Moreover, they are incompatible with the linear bivectors $P'_{1,2}$ (3.7) as well. This means that we have different bi-Hamiltonian structures associated with the Lagrange top. This fact deserves further investigation.

4. Another integrable system on $e^*(3)$

4.1. The Goryachev–Chaplygin top

The well-known Goryachev–Chaplygin case in rigid body dynamics is described by the following integrals of motion:

$$H_1 = J_1^2 + J_2^2 + 4J_3^2 + ax_1, \qquad H_2 = 2\left(J_1^2 + J_2^2\right)J_3 - ax_3J_1, \qquad a \in \mathbb{R}. \tag{4.1}$$

On the fixed level $(x, J) = 0$ of the second Casimir function the Hamilton function H_1 commutes with an additional cubic integral of motion H_2. This fact ensures the integrability of the Goryachev–Chaplygin case.

Substituting ansatz (3.10) into (2.12) at $f_1 = 0$ one gets the following solution,

$$f_2 = 0, \qquad f_3 = -1,$$

and

$$g_1 = -\frac{J_1J_3}{(x\wedge J)_3}, \qquad g_2 = -\frac{J_2J_3}{(x\wedge J)_3}, \qquad g_3 = \frac{J_1^2 + J_2^2}{(x\wedge J)_3} \tag{4.2}$$

The corresponding polynomial P^f (3.10) has been obtained in [18] by using the r-matrix formalism and the Sklyanin brackets.

Remark 6. The same bivector P^f could be easily found by using the Liouville vector field X with polynomial entries:

$$P^f = \mathcal{L}_X(P_0), \qquad X = \sum X_m(z)\partial/\partial z_m, \qquad z = (x, J).$$

If we suppose that $X_m(z)$ are arbitrary quadratic polynomials on M

$$X_m = \sum_{ij}^{n} c_m^{ij} z_i z_j, \qquad m = 2, \ldots, 6, \qquad c_{ij}^m \in \mathbb{C}, \tag{4.3}$$

then from (2.12) one easily gets

$$X = \big(0, \quad x_3 J_2, \quad -x_2 J_2, \quad -J_1 J_3, \quad 0, \quad -J_2^2 - J_3^2\big)\,. \tag{4.4}$$

In contrast to rational functions (4.2) here we have simple polynomials only. As a useful by-product we directly prove that the bivector P_f is 2-coboundary in the corresponding Poisson–Lichnerowicz cohomology.

In this case the control matrix F is equal to

$$F = \begin{pmatrix} 2J_3 & -1 \\ -J_1^2 - J_2^2 & 0 \end{pmatrix} \tag{4.5}$$

and its eigenvalues

$$q_{1,2} = J_3 \pm \sqrt{J_1^2 + J_2^2 + J_3^2} \tag{4.6}$$

satisfy the following dynamical equations,

$$(-1)^j (q_1 - q_2)\dot{q_j} = 2\sqrt{\mathcal{P}(q_j)^2 - |x|^2 a^2 q_j^2}, \qquad \mathcal{P}(\lambda) = \lambda^3 - \lambda H_1 + H_2. \tag{4.7}$$

These equations are reduced to the Abel–Jacobi equations and, therefore, they are solved in quadratures.

4.2. The Sokolov system on the sphere

Let us consider another integrable at $(x, J) = 0$ system on $e^*(3)$ [14] with integrals of motion second and fourth orders:

$$\begin{aligned} H_1 &= J_1^2 + J_2^2 + 2J_3^2 + a(x_3 J_1 - J_3 x_1) + 2bJ_3, \\ H_2 &= \big(J_1^2 + J_2^2 + J_3^2\big)(2J_3 + 2b - ax_1)^2. \end{aligned} \tag{4.8}$$

Using the same anzats (4.3) for the components $X_m(z)$ of the Liouville vector field as for the Goryachev–Chaplygin top one gets the same solution (4.4) of the equations (2.12). In this case the control matrix reads as

$$F = \begin{pmatrix} J_3 & \frac{1}{2(2J_3+2b-ax_1)} \\ 2\big(J_1^2 + J_2^2 + J_3^2\big)(2J_3 + 2b - ax_1) & J_3 \end{pmatrix}. \tag{4.9}$$

Its eigenvalues coincide with the Chaplygin variables (4.6), which are also the separated variables for the Sokolov system.

4.3. The Kowalevski top

Let us consider the Kowalevski top with the following integrals of motion:

$$\begin{aligned} H_1 &= J_1^2 + J_2^2 + 2J_3^2 - 2bx_1, \\ H_2 &= ((J_1 + \mathrm{i}J_2)^2 + 2b(x_1 + \mathrm{i}x_2))((J_1 - \mathrm{i}J_2)^2 + 2b(x_1 - \mathrm{i}x_2)). \end{aligned} \tag{4.10}$$

The solution of equations (2.12) has been constructed in [19] by using the r-matrix formalism and the reflection equation algebra. In our notations this solution is defined by

$$\begin{aligned} f_1 &= -2J_1 - \frac{(2x_1 J_2 - x_2 J_1)\big(b(x_2 J_2 + x_3 J_3) + J_1 J_3^2\big)}{J_2^2 (x\wedge J)_3}, \\ f_2 &= J_2 - \frac{J_1^2 + bx_1}{J_2} - \frac{x_1\big(2J_3^2 - bx_1\big)}{(x\wedge J)_3} + \frac{x_2 J_3 (2J_1 J_3 + bx_3)}{J_2 (x\wedge J)_3} + \frac{x_1 (J_1 J_3 + bx_3)^2}{J_2^2 (x\wedge J)_3}, \\ f_3 &= J_3 - \frac{bx_2^2 J_3}{J_2 (x\wedge J)_3} + \frac{(J_1 J_3 + bx_3)(J_1 (x\wedge J)_3 - bx_1 x_2)}{J_2^2 (x\wedge J)_3} \end{aligned}$$

and

$$
\begin{aligned}
g_1 &= \frac{b(x_1J_1 + x_3J_3)}{(x\wedge J)_3} + \frac{bx_2\left(J_1^2 - J_3^2 + bx_1\right)}{J_2(x\wedge J)_3} + \frac{J_1(J_1J_3 + bx_3)^2}{J_2^2(x\wedge J)_3},\\
g_2 &= \frac{2bx_2J_1}{(x\wedge J)_3} + \frac{b^2x_2^2 + J_1J_3(J_1J_3 + bx_3)}{J_2(x\wedge J)_3} + \frac{bx_2J_3(J_1J_3 + bx_3)}{J_2^2(x\wedge J)_3},\\
g_3 &= \frac{bx_3J_1}{(x\wedge J)_3} + \frac{bx_2(J_1J_3 + bx_3)}{J_2(x\wedge J)_3} + \frac{J_3(J_1J_3 + bx_3)^2}{J_2^2(x\wedge J)_3}.
\end{aligned}
$$

The corresponding separated variables $q_{1,2}$ are the famous Kowalevski variables [19]. In these variables bivectors P and P^f have the form (3.12). It allows us to prove that the second bivector P_f is the 2-coboundary in the corresponding Poisson–Lichnerowicz cohomology.

5. Concluding remarks

The main result in this paper is construction of the different bi-Hamiltonian structures for the Lagrange top, which have the same foliation by symplectic leaves. The corresponding three incompatible Poisson bivectors may be associated with the 2-coboundaries in the Poisson–Lichnerowicz cohomology defined by canonical bivector P on $e^*(3)$.

As a last remark, we observe that similar bi-Hamiltonian structures exist for some other integrable systems on $e^*(3)$, for instance for the Kowalevski top. The similar 2-coboundaries in the Poisson–Lichnerowicz cohomology on $so^*(4)$ were considered in [20, 21].

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